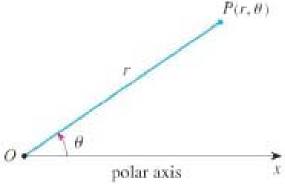
Polar Co-ordinates

A polar coordinate system, gives the co-ordinates of a point with reference to a point O and a half line or ray starting at the point O. We will look at polar coordinates for points in the xy-plane, using the origin $(0, 0)$ and the positive x-axis for reference.

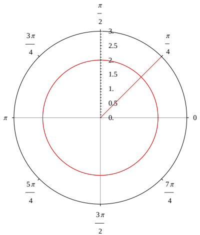


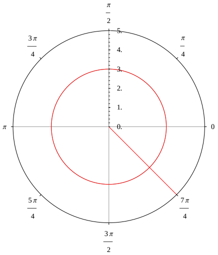
A point P in the plane, has polar coordinates (r, θ), where r is the distance of the point from the origin and θ is the angle that the ray $|OP|$ makes with the positive x-axis.

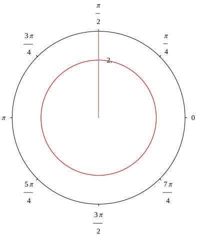
Example 1

Example 1 Plot the points whose polar coordinates are given by

$$(2, \frac{\pi}{4}) \quad (3, -\frac{\pi}{4}) \quad (3, \frac{7\pi}{4}) \quad (2, \frac{5\pi}{2})$$





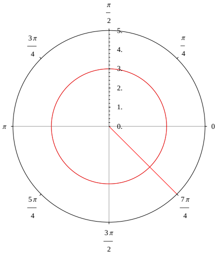


Example 1

Note the representation of a point in polar coordinates is not unique. For instance for any θ the point $(0, \theta)$ represents the pole O. We extend the meaning of polar coordinate to the case when r is negative by agreeing that the two points (r, θ) and $(-r, \theta)$ are in the same line through O and at the same distance $|r|$ but on opposite side of O. Thus

$$(-r, \theta) = (r, \theta + \pi)$$

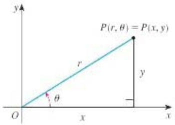
Example 2 Plot the point $(-3, \frac{3\pi}{4})$



Polar to Cartesian coordinates

To convert from Polar to Cartesian coordinates, we use the identities:

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta}$$



Example 3 Convert the following (given in polar co-ordinates) to Cartesian coordinates $(2, \frac{\pi}{4})$ and $(3, -\frac{\pi}{3})$

- For $(2, \frac{\pi}{4})$, we have $r = 2$, $\theta = \frac{\pi}{4}$. In Cartesian co-ordinates, we get $x = r\cos\theta = 2\cos(\pi/4) = 2\frac{1}{\sqrt{2}} = \sqrt{2}$
  we get $y = r\sin\theta = 2\sin(\pi/4) = 2\frac{1}{\sqrt{2}} = \sqrt{2}$
- For $(3, -\frac{\pi}{3})$, we have $r = 3$, $\theta = -\frac{\pi}{3}$. In Cartesian co-ordinates, we get $x = r\cos\theta = 3\cos(-\frac{\pi}{3}) = 3\frac{1}{2} = \frac{3}{2}$
  we get $y = r\sin\theta = 3\sin(-\frac{\pi}{3}) = 3\frac{-\sqrt{3}}{2} = \frac{-3\sqrt{3}}{2}$

Cartesian to Polar coordinates

To convert from Cartesian to polar coordinates, we use the following identities

$$\boxed{r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}}$$

When choosing the value of θ, we must be careful to consider which quadrant the point is in, since for any given number a, there are two angles with $\tan\theta = a$, in the interval $0 \leq \theta \leq 2\pi$.

Example 3 Give polar coordinates for the points (given in Cartesian co-ordinates) $(2, 2)$, $(1, -\sqrt{3})$, and $(-1, \sqrt{3})$

- For $(2, 2)$, we have $x = 2$, $y = 2$. Therefore $r^2 = x^2 + y^2 = 4 + 4 = 8$, and $r = \sqrt{8}$.
  We have $\tan\theta = \frac{y}{x} = 2/2 = 1$.
  Since this point is in the first quadrant, we have $\theta = \frac{\pi}{4}$.
  Therefore the polar co-ordinates for the point $(2, 2)$ are $(\sqrt{8}, \frac{\pi}{4})$.
- For $(1, -\sqrt{3})$, we have $x = 1$, $y = -\sqrt{3}$. Therefore $r^2 = x^2 + y^2 = 1 + 3 = 4$, and $r = 2$.
  We have $\tan\theta = \frac{y}{x} = -\sqrt{3}$.
  Since this point is in the fourth quadrant, we have $\theta = \frac{-\pi}{3}$.
  Therefore the polar co-ordinates for the point $(1, -\sqrt{3})$ are $(2, \frac{-\pi}{3})$.

Example 3

Example 3 Give polar coordinates for the point (given in Cartesian co-ordinates) $(-1, \sqrt{3})$

- For $(-1, \sqrt{3})$, we have $x = -1$, $y = \sqrt{3}$. Therefore $r^2 = x^2 + y^2 = 1 + 3 = 4$, and $r = 2$.
  We have $\tan\theta = \frac{y}{x} = -\sqrt{3}$.
  Since this point is in the second quadrant, we have $\theta = \frac{2\pi}{3}$.
  Therefore the polar co-ordinates for the point $(-1, \sqrt{3})$ are $(2, \frac{2\pi}{3})$.

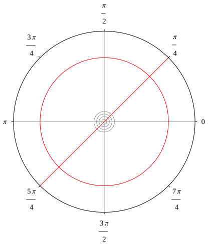
Graphing Equations in Polar Coordinates

The graph of an equation in polar coordinates $r = f(\theta)$ or $F(r, \theta) = 0$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

- **Lines:** A line through the origin $(0, 0)$ has equation $\theta = \theta_0$
- **Circle centered at the origin:** A circle of radius r_0 centered at the origin has equation $r = r_0$ in polar coordinates.

Example 4 Graph the following equations $r = 5$, $\theta = \frac{\pi}{4}$

- The equation $r = 5$ describes a circle of radius 5 centered at the origin. The equation $\theta = \frac{\pi}{4}$ describes a line through the origin making an angle of $\frac{\pi}{4}$ with the positive x axis.



Example 5

Example 5 Graph the equation $r = 6\sin\theta$ and convert the equation to an equation in Cartesian coordinates.

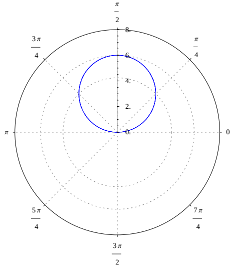
θ	r
0	
$\frac{\pi}{4}$	
$\frac{\pi}{2}$	
$\frac{3\pi}{4}$	
π	
$\frac{5\pi}{4}$	
2π	

- We find the value of r for specific values of θ in order to plot some points on the curve. We note that it is enough to sketch the graph on the interval $0 \leq \theta \leq 2\pi$, since $(r(\theta), \theta) = (r(\theta + 2\pi), \theta + 2\pi)$.

Example 5 Graph the equation $r = 6\sin\theta$ and convert the equation to an equation in Cartesian coordinates.

θ	r
0	0
$\frac{\pi}{4}$	$6/\sqrt{2}$
$\frac{\pi}{2}$	6
$\frac{3\pi}{4}$	$6/\sqrt{2}$
π	0
$\frac{5\pi}{4}$	$-6/\sqrt{2}$
$\frac{3\pi}{2}$	-6

- We find the value of r for specific values of θ in order to plot some points on the curve. We note that it is enough to sketch the graph on the interval $0 \leq \theta \leq 2\pi$, since $(r(\theta), \theta) = (r(\theta + 2\pi), \theta + 2\pi)$.
- When we plot the points, we see that they lie on a circle of radius 3 centered at $(0, 3)$ and that the curve traces out this circle twice in an anticlockwise direction as θ increases from 0 to 2π.

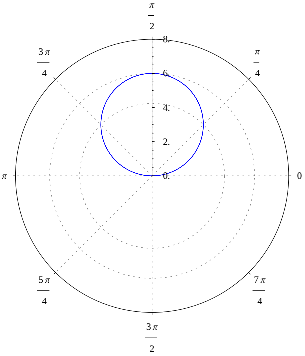


- The equation in Cart. co-ords is $x^2 + (y - 3)^2 = 9$.

Example 5

Example 5 Graph the equation $r = 6\sin\theta$ and convert the equation to an equation in Cartesian coordinates.

θ	r	Cartesian Co-ords (x, y)
0	0	$(0, 0)$
$\frac{\pi}{4}$	$6/\sqrt{2}$	$(3, 3)$
$\frac{\pi}{2}$	6	$(0, 6)$
$\frac{3\pi}{4}$	$6/\sqrt{2}$	$(-3, 3)$
π	0	$(0, 0)$
$\frac{5\pi}{4}$	$-6/\sqrt{2}$	$(3, 3)$
$\frac{3\pi}{2}$	-6	$(0, 6)$



It may help to calculate the cartesian co-ordinates in order to sketch the curve.

Example 6

Example 6 Graph the equation $r = 1 + \cos\theta$. Check the variations shown at end of lecture notes.

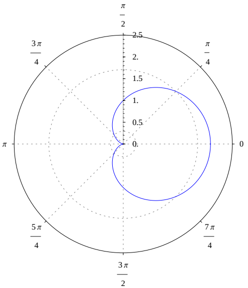
θ	r	Cartesian Co-ords (x, y)
0		
$\frac{\pi}{4}$		
$\frac{\pi}{2}$		
$\frac{3\pi}{4}$		
π		
$\frac{5\pi}{4}$		
$\frac{3\pi}{2}$		
$\frac{7\pi}{4}$		
2π		

Example 6

Example 6 Graph the equation $r = 1 + \cos\theta$. Check the variations shown at end of lecture notes.

θ	r	Cartesian Co-ords (x, y)
0	2	$(2, 0)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}+1}{\sqrt{2}}$	$(\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2})$
$\frac{\pi}{2}$	1	$(0, 1)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}-1}{\sqrt{2}}$	$(\frac{-(\sqrt{2}-1)}{2}, \frac{\sqrt{2}-1}{2})$
π	0	$(0, 0)$
$\frac{5\pi}{4}$	$\frac{\sqrt{2}-1}{\sqrt{2}}$	$(\frac{-(\sqrt{2}-1)}{2}, \frac{-(\sqrt{2}-1)}{2})$
$\frac{3\pi}{2}$	1	$(0, -1)$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}+1}{\sqrt{2}}$	$(\frac{(\sqrt{2}+1)}{2}, \frac{-(\sqrt{2}+1)}{2})$
2π	2	$(2, 0)$

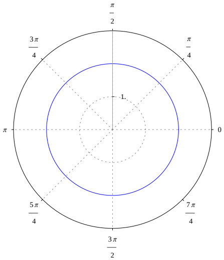
- We find the value of r for specific values of θ in order to plot some points on the curve. We note that it is enough to sketch the graph on the interval $0 \le \theta \le 2\pi$, since $(r(\theta), \theta) = (r(\theta + 2\pi), \theta + 2\pi)$.
- When we plot the points, we see that they lie on a heart shaped curve.

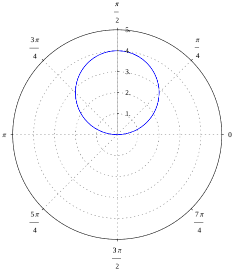


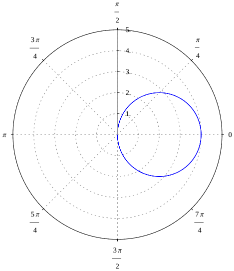
Circles

We have many equations of circles with polar coordinates: $r = a$ is the circle centered at the origin of radius a, $r = 2a\sin\theta$ is the circle of radius a centered at $(a, \frac{\pi}{2})$ (on the y-axis), and $r = 2a\cos\theta$ is the circle of radius a centered at $(a, 0)$ (on the x-axis).

Below, we show the graphs of $r = 2$, $r = 4\sin\theta$ and $r = 4\cos\theta$.







Tangents to Polar Curves

If we want to find the equation of a tangent line to a curve of the form $r = f(\theta)$, we write the equation of the curve in parametric form, using the parameter θ.

$$\boxed{x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta}$$

From the calculus of parametric equations, we know that if f is differentiable and continuous we have the formula:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Note As usual, we locate horizontal tangents by identifying the points where $dy/dx = 0$ and we locate vertical tangents by identifying the points where $dy/dx = \infty$

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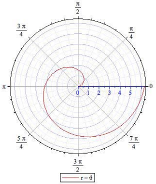
Example: Tangents to Polar Curves

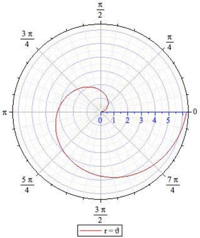
Example 8 Find the equation of the tangent to the curve $r = \theta$ when $\theta = \frac{\pi}{2}$

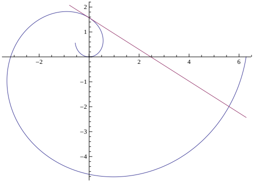


- We have parametric equations $x = \theta \cos\theta$ and $y = \theta \sin\theta$.
- $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}$
- When $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{1+0}{0-\frac{\pi}{2}} = -\frac{2}{\pi}$.
- When $\theta = \frac{\pi}{2}$, the corresponding point on the curve in polar co-ordinates is given by $(\frac{\pi}{2}, \frac{\pi}{2})$ and in Cartesian co-ordinates by $(0, \frac{\pi}{2})$.

Example 8

Find the equation of the tangent to the curve $r = \theta$ when $\theta = \frac{\pi}{2}$





- We have parametric equations $x = \theta \cos\theta$ and $y = \theta \sin\theta$. and $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}$
- When $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{1+0}{0-\frac{\pi}{2}} = -\frac{2}{\pi}$.
- When $\theta = \frac{\pi}{2}$, the corresponding point on the curve in polar co-ordinates is given by $(\frac{\pi}{2}, \frac{\pi}{2})$ and in Cartesian co-ordinates by $(0, \frac{\pi}{2})$.
- Therefore the equation of the tangent line, when $\theta = \frac{\pi}{2}$ is given by $(y - \frac{\pi}{2}) = -\frac{2}{\pi}x$.